A Controller Design of Linear Quadratic by Regulator, Integral, and Gaussian

Kanchit Pawananont [1], Thawikan Saneewong Na Ayuthaya[1], Piyapat Poungsri[2] and Kritchanan Charoensuk[1]*

[1]Department of Industrial Technology and Innovation Management, Faculty of Science and Technology, Pathumwan Institute of Technology, Thailand

[2]Department of Electrical Engineering, Faculty of Engineering, Pathumwan Institute of Technology, Thailand

Abstract: The paper aims to outline the key design principles of the aero pendulum balancing system and its three related controllers: LQR, LQI, and LQG. The design incorporates dynamic principles and Newton's laws, leading to a non-linear mathematical model that must be converted into a linear form for control design. The LQR controller is selected as the central controller, serving as a foundation for developing the LQI and LQG controllers to guarantee stability and eliminate interference within the Aero pendulum system. The MATLAB Simulink program is employed due to its smooth integration with the Arduino board and Aero pendulum, allowing for simulation and experimentation. The controller logic utilizes the system's features to stabilize the Aero pendulum and uphold the desired angle through simulation. The article assesses the controller design based on the Aero pendulum's inherent structure and angular performance, contrasting it with the design derived from its mathematical model and analyzing the performance index of the control system. In conclusion, the article highlights the use of LQR, LQI, and LQ controllers for stabilizing the Aero pendulum, underscoring the efficacy of the linear equation from the mathematical model in managing and stabilizing the system.

***Keywords*— Arduino, Simulink, LQR, LQI, LQG**

I. Introduction

Feedback control encompasses both traditional and modern control methodologies. Conventional control techniques, such as those utilizing Proportional-Integral-Derivative (PID) controllers [1-2], typically involve deriving a mathematical model represented by differential equations. This model is then transformed from the time domain to the frequency domain and illustrated using a block diagram. Conventional control approaches generally operate under a single-input-single-output (SISO) framework. System stability is evaluated through methods like Root Locus in system analysis, which leads to the controller's gain value design.

In contrast, modern control methods allow for the establishment of mathematical models without needing a transformation to the frequency domain. Control rules, such as pole placement or designs involving Linear Quadratic Regulator (LQR), Linear Quadratic Integrator (LQI), and Linear Quadratic Gaussian (LQG) controllers, can be determined based on stage specifications [2-4]. This method may result in additional variables; for instance, in the case of a second-order system, two state variables—X1 and X2—would necessitate the use of two sensors for measurement. However, implementing state estimation techniques, like the Kalman filter [5], allows for estimating system states using only a mathematical model alongside the available measured states. This can potentially reduce the necessity for measuring both states, contingent on verifying system observability.

This article explores the control of an aero pendulum utilizing LQR, LQI, and LQG controllers to enhance the control performance and stability of the system.

II. Modeling of the Aero pendulum

The control developer affords a condensed and low-cost testing stage for the practical demonstration of the feedback control concept that reasons are agreed well with the report [13]. In this research, the aero-pendulum of angle position control and disturbance reimbursement approach was created for an aero-pendulum using the soft computing paradigm of Simulink. The pendulum arm is rotated via the thrust generated by one axial contra-rotating powered propeller installed at its free end, shown in Figure 1. The propeller assembly, shown in Figure 1, consists of axial rotors installed on concentric shafts.

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*Corresponding author: Kritchanan Charoensuk, 1Department of Industrial Technology and Innovation Management, Faculty of Science and Technology, Pathumwan Institute of Technology, Bangkok 10330, Thailand (E-mail: kritchanan.c@pit.ac.th)

$$\frac{\theta(s)}{T(\mathrm{s})} = \frac{1}{js^2 + cs + m_L gd} \quad (1)$$

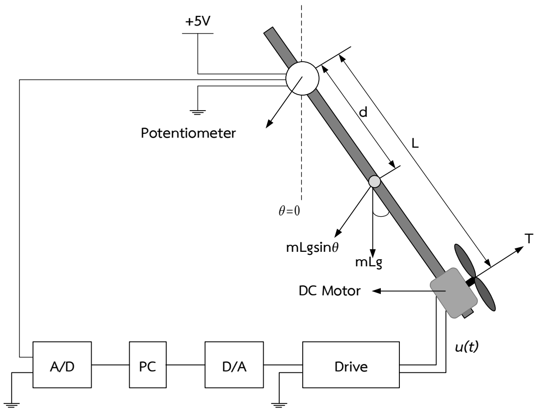

Fig. 1 Free body diagrams of aero-pendulum

The rational equation of motion and voltage applied to a DC motor can thus be written as follows:

$$T(s) = K_m \cdot V(s) \quad (2)$$

From Eq. (1), The pendulum gets the stability. Then, one can obtain $\frac{d}{dt}\theta(t) = 0$ and $\frac{d^2}{dt^2}\theta(t) = 0$. In this way, Eq. (2) can be written as

$$T = m_L \cdot g \cdot d \cdot \sin\theta \quad (3)$$

Where θ is the angle at the stable situation. Thus, substituting Eq. (2) in Eq.(3) leads to

$$K_m = \frac{m_L gd \sin\theta}{V} \quad (4)$$

Fig. 2 Experiment kit of aero-pendulum

Finally, from Eq. (1) and Eq. (2), the open-loop block diagram of the pendulum system can be obtained as follows:

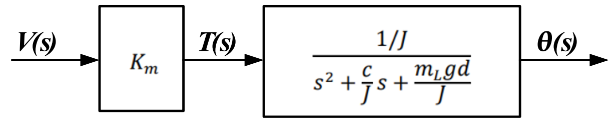

Fig. 3 Open-loop diagram of suspended pendulum

Now, from the above figure, the transfer function of the suspended pendulum can be written as follows:

$$\frac{\theta(s)}{V(s)} = \frac{\frac{K_m}{J}}{S^2 + \frac{c}{J}S + \frac{m_L gd}{J}} \quad (5)$$

The design features a robust steel base measuring 3 mm in thickness, 4 cm in width, and 60 cm in length, weighing approximately 0.22 kg. This base is anchored to an acrylic column at the center using bolts, ensuring stability for the test setup during operation. The system is connected to the Aero Pendulum axis via a variable resistor, providing precise control over its movement. Attached to the system is a 12-volt DC motor, which incorporates a propeller for functionality. The entire assembly is designed to move along the Aero Pendulum axis, with one end secured and the other held fast by a dedicated fastener. The spindle used in this setup has dimensions of 3 mm in thickness, 1.5 cm in width, and 12.7 cm in length. For the resistance circuit control, a resistor with a value of ten kΩ is employed, featuring a 0.5% error margin and 0.5% linearity. This component is critical for adjusting the Aero Pendulum axis accurately. In response to thrust torque, the pendulum's angular motion is governed by a second-order differential equation. The derived equation of motion has led to the formulation of Root Locus models in the time domain. Parameter estimations have been conducted, resulting in values detailed in Table 1 for the Aero Pendulum model.

TABLE I
PARAMETER CONSTANTS FOR AERO PENDULUM MODEL

Parameter	Values
K	7.65 m/s
L	0.23 m
m	0.22 kg
d	0.015 m
J	0.51 kg.m^2
g	9.81 m/s^2
c	0.0027 kg.m^2/s

Upon computing the transfer function in accordance with Equation 3 derived from the Aero Pendulum model, the following results are obtained:

$$G(s) = \frac{\theta(s)}{V(s)} = \frac{15}{S^2 + 0.005S + 0.063} \quad (6)$$

III. CONTROLLER DESIGN

The Aero pendulum controller design theory comprises three parts. The first part delves into the LQR controller design theory, known for its robustness to system parameter changes and ability to eliminate incoming disturbances. The second section elaborates on the LQI controller design theory. The model showcases its increased robustness to system parameter variations and its ability to enhance the integral to eliminate noise with unit gain entering the system. Lastly, the third section explores the theory of LQG controller design, highlighting its robustness to system parameter changes by incorporating a feedback observer to estimate the state and its capacity to eliminate incoming noise.

A. LQR controller

The LQR controller design [6] is a control methodology that relies on feedback from the state signal within the Aero pendulum control system. The control signal's gain is computed, as depicted in Figure 4. In the LQR (Linear Quadratic Regulator) design process, identifying closed-loop characteristics that align with our design objectives is crucial for determining the optimal gain matrix (K). It is important to evaluate the performance of the system relative to the effort required to achieve that performance, as this relationship can significantly influence the effectiveness of the control strategy.

The linear quadratic regulator (LQR) theory is primarily formulated for linear mathematical models. In the existing literature, several methodologies are available for determining a linearized mathematical model, including identification techniques, linearization methods, and the State-Dependent Riccati Equation (SDRE) approach. It is important to note that the SDRE method applies only to a specific type of nonlinear system model.

Typically, the process of nonlinear optimal control involves employing Jacobian linearization around various operating points of the model system. In this context, this paper assumes that the nonlinear mathematical model of the system under study is well-defined. By linearizing this model at several operating points (denoted as k), we can derive a range of more straightforward and localized mathematical representations of the system. This approach facilitates a better understanding and implementation of optimal control strategies for the nonlinear system.

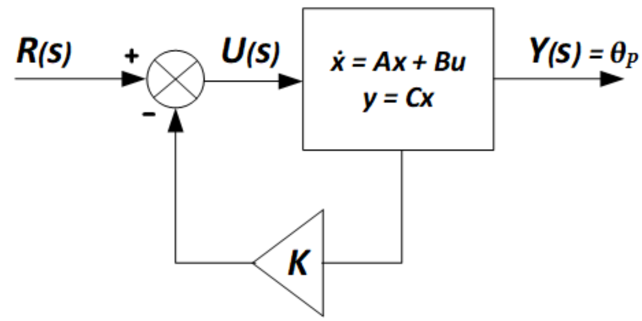

Fig. 4 Block diagram LQR control system.

The design aims to minimize the performance index value J (performance index) to the most significant period possible.

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt \quad (7)$$

In this scenario, Q is a positive quasi-definite symmetric matrix used as a significant weighting matrix in regulating individual state variables. On the other hand, R is a positive-definite symmetric matrix serving as a weight for the control signal without any constraints. Determining the optimal K-value for this system allows for the computation of an optimal control signal.

$$u(t) = -Kx(t) = -R^{-1}B^T P x(t) \quad (8)$$

For the matrix P can reduced as follow:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (9)$$

When designing the controller based on Eq. 9, it is essential to calculate the optimal value of matrix P for stability. The system's positive symmetric nature is crucial for stability to verify stability, and it is necessary to confirm that the A-BK matrix is stable and then substitute matrix P to derive the most suitable K value.

B. LQI controller

The LQI (Linear Quadratic Integral) controller design is an advanced control method employed in the Aero pendulum control system, utilizing feedback from status signals. In this design, the control signal gain is systematically calculated, and an integral component is incorporated to mitigate interference caused by the unit development rate entering the system, as illustrated in Figure 5.

The LQI control method significantly broadens the range of states that influence the control input, denoted as R(s). By integrating additional dimensions into the control design, this approach enhances the reference tracking by correlating directly with the number of outputs included in the system. Examining the theoretical principles underpinning LQI control can help one gain a deeper understanding of this phenomenon.

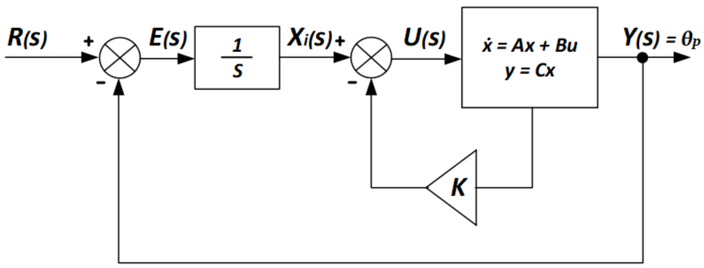

Fig. 5 Block diagram LQI control system.

The signal control can represent as follow Eq.10:

$$u = -Kz = -K\begin{bmatrix} x & x_i \end{bmatrix} \quad (10)$$

The control law shown in (10) is designed to minimize the cost function below (assuming a reference value of zero)

$$J(u) = \int_0^\infty \{z^T Qz + u^T Ru + 2z^T Nu\} dt \quad (11)$$

The current study involves the controller's tracking operation for two output variables. The x_i vector contains two members representing the errors for the respective terms. Additionally, the vector z consists of seven members, which comprise the original state variables and the two error terms:

$$z = \begin{bmatrix} n & p_3^* & T_3^* & p_6^* & T_6^* & e_n & e_{EPR} \end{bmatrix} \quad (12)$$

To find e_n, subtract the reference rotor speed from the actual rotor speed, and for e_{EPR}, calculate the difference between their respective EPR values.

C. LQG controller

In the LQG controller design[8], a control methodology utilizes the feedback principle of state signals within the Aero pendulum control system to calculate the control signal's gain. A feedback observer design, such as the Kalman filter, assesses the state and reduces the impact of noise entering the system. This process is illustrated in Figure 6. The linear quadratic Gaussian (LQG) control scheme is an effective method for managing output measurements corrupted by Gaussian noise and can be applied to control perturbed non-linear systems. This approach is characterized by its simplicity in design and ease of implementation, requiring a minimal number of design parameters. Moreover, it is computationally and algorithmically less complex, as it addresses a minimizing quadratic cost function to derive an optimal state feedback law.

The LQG framework represents an optimal stochastic control design problem, integrating the Linear Quadratic Regulator (LQR) principles and the Kalman filter for comprehensive state feedback and state estimation. Applying the separation principle in LQG control design facilitates implementation, enabling the independent design of the state estimator and the feedback controller.

A crucial aspect of the LQG design process is the individual design or acquisition of the optimal state feedback (K_lqr) and state estimation (K_se) gains associated with the LQR and Kalman filter, contingent upon fulfilling the controllability (or stabilizability) and observability (or detectability) criteria. Fulfilling these criteria is essential for the successful resolution of the Riccati equations governing optimal state feedback and state estimation gains.

In scenarios where certain system states may be unobservable, as often encountered in real-world control design applications, the Kalman filter can be utilized for complete state estimation, provided that the system's available measurements contain critical information regarding the system states. The Kalman filter functions as a low-pass filter, demonstrating efficacy in rejecting disturbances within the system. The control input and system output are the primary inputs to the Kalman filter.

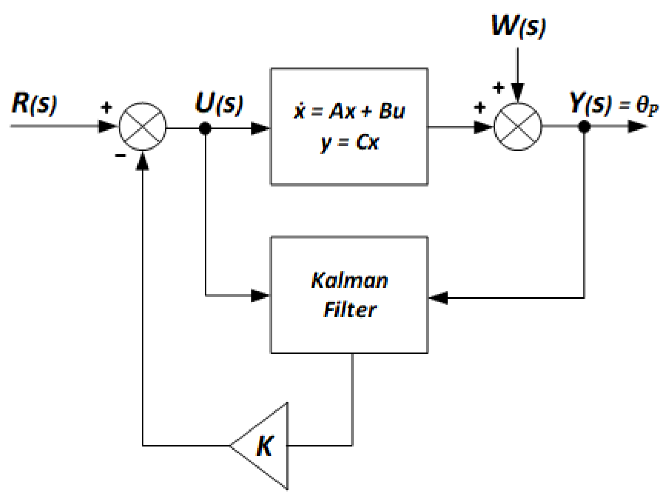

Fig. 6 Block diagram LQG control system.

The conventional structure of the Kalman filter typically includes a set of iterative equations. The system's discrete model P and the system covariance matrix, along with L, the Kalman gain matrix, are updated continuously through these iterative equations (13). L can be derived from this set of iterative equations. These iterative equations are initiated with an initial covariance matrix $P(k/k)$.

$$P(k+1/k) = A(T)P(k/k)A^T(t) + C_d(T)Q_e C_d^T(T) \quad (13)$$

$$L(k+1/k) = P(k+1/k)C^T(T)\{C(T)P(k+1/k)C^T(T)+\mathrm{R_e}\}^{-1} \quad (14)$$

$$P(k+1/k+1) = \{I - L(k+1)C(T)\}P(k+1/\mathrm{k}) \quad (15)$$

The procedure continues by reintroducing the covariance $P(k + 1/k +1)$, (15), into equation (13) as $P(k/k)$ until $L(k +1)$ stabilizes. In this context, $C_d(T)$ represents the disturbance transition, R_e stands for measurement noise covariance, and Q_e denotes disturbance noise covariance matrices. Suppose $P(k/k) = P_1$, $P(k +1/k + 1) = P_3$, and $P(k + 1/k) = P_2$, then equations (13) – (15) are transformed into:

$$P_2 = AP_1A^T + C_dQ_eC_d^T \quad (16)$$

$$L = P_2C^T\{CP_2C^T + R_e\}^{-1} \quad (17)$$

$$P_3 = \{I - LC\}P_2 \quad (18)$$

IV. Experiment

The discussion involves a computer-based simulation of the Aero pendulum using a mathematical model to establish the controller and assess its stability [9]. The assessment is divided into two components. The initial segment focuses on the computer-based simulation of the Aero pendulum. At the same time, the subsequent part entails the simulation and experimentation of the LQR, LQI, and LQG controllers used to regulate the Aero pendulum.

A. The simulation of an Aero pendulum via computers.

The analysis of the Aero pendulum through computer simulation involves determining the moment of inertia by continuously adjusting the lower equilibrium point of the control system until the desired value is achieved. This is accompanied by introducing a noise signal into the system using a pulse signal. The testing of the Aero pendulum involves a 10-second noise signal to ascertain its open-loop oscillation response. Natural structures and mathematical models are then utilized to create a computer model of the Aero pendulum, as illustrated in Figure 7.

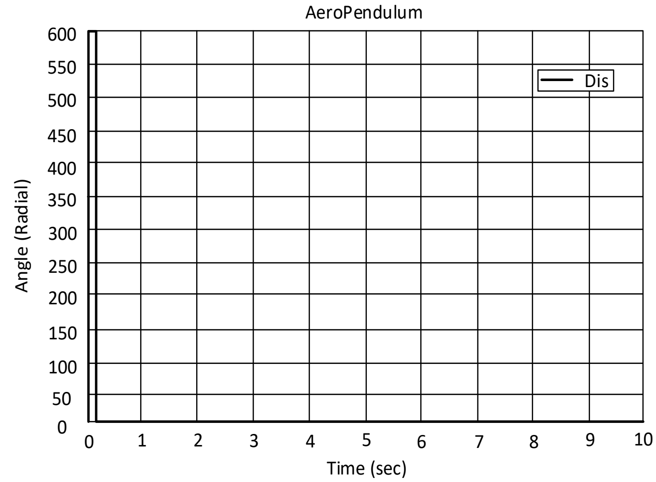

Fig. 7 Angle for Aero pendulum with disturbance signal

B. Experimenting with controlling the Aero pendulum With LQR.

In this section, we will explore the application of an LQR controller to maintain the stability of the Aero pendulum. The LQR controller has been designed and tested with the Aero pendulum structure. We will discuss the computer simulation and control of the actual structure to ensure the Aero pendulum's stability. Additionally, we will introduce pulse noise into the Aero pendulum control system to observe its effects, as depicted in Figure 8. Furthermore, we will showcase the application of an LQR controller to stabilize the Aero pendulum at the equilibrium position, as illustrated in Figure 9.

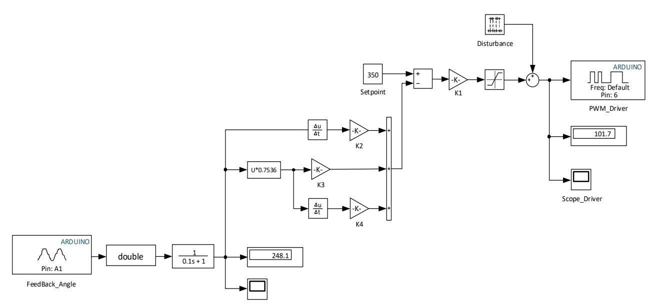

Fig. 8 Program for Simulink LQR controller.

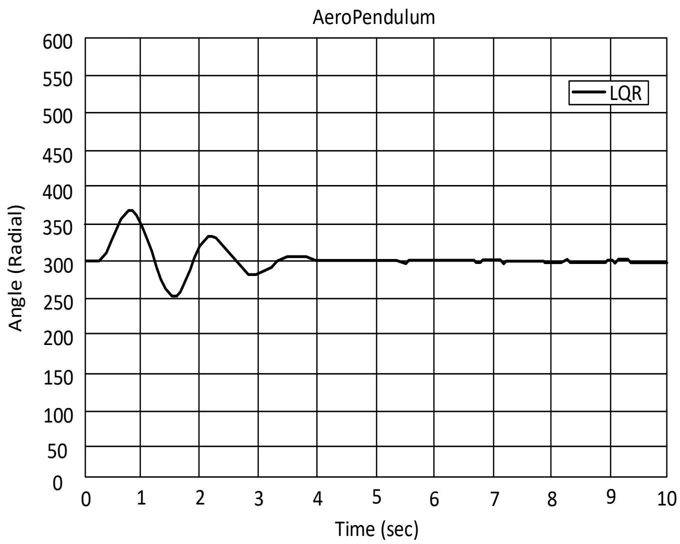

Fig. 9 Angle for Aero pendulum with LQR controller.

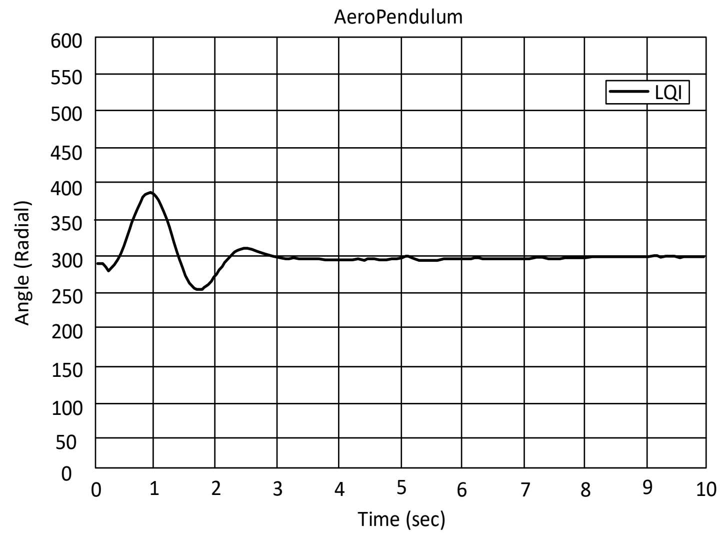

Fig. 11 Angle for Aero Pendulum with LQI controller.

C. Experimenting with controlling the Aero pendulum With LQI.

In this segment, we will be focusing on simulating control mechanisms to ensure the Aero pendulum's stability using the LQI controller derived from the design and practical assessment of the Aero pendulum system[10]. The discussion will encompass the computer simulation and control of the physical structure to maintain the stability of the Aero pendulum. Furthermore, the introduction of pulse noise into the Aero pendulum control system, leading to instability, will be addressed. The block diagram in Figure 10 and the noise depicted in Figure 7 will be referenced, culminating in Figure 11, which illustrates the outcome of utilizing an LQI controller to stabilize the Aero pendulum at the equilibrium position.

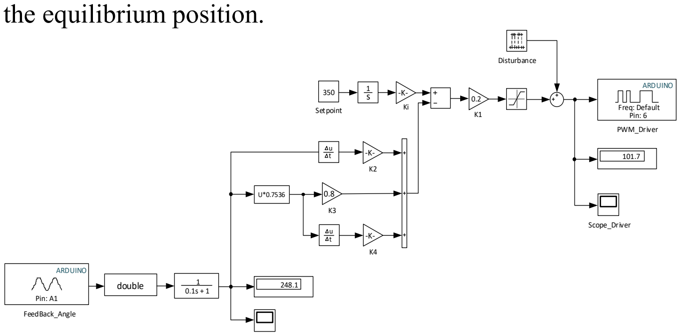

Fig. 10 Program for Simulink LQI controller.

D. Experimenting with controlling the Aero pendulum With LQG.

In the following topic, we will explore simulating control methods to ensure the stability of the Aero pendulum using the LQG controller[11,12]. The controller is designed based on experimentation with the Aero pendulum structure. We will cover computer simulations and the control of the physical structure to maintain the stability of the Aero pendulum. Additionally, we will introduce pulse noise into the Aero pendulum control system to intentionally destabilize it. Figure 1 2 illustrates the block diagram, while Figure 7 displays the introduced noise. Furthermore, Figure 13 depicts the Aero pendulum control system utilizing an LQG controller to stabilize the pendulum at its equilibrium position

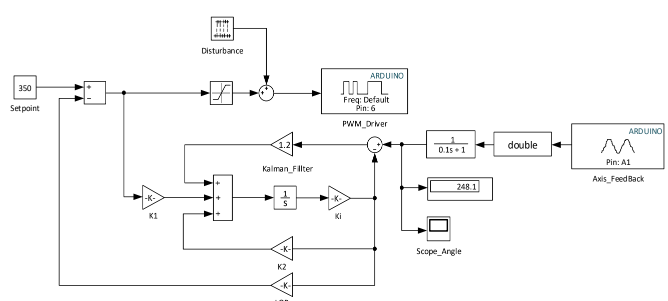

Fig. 12 Program for Simulink LQG controller.

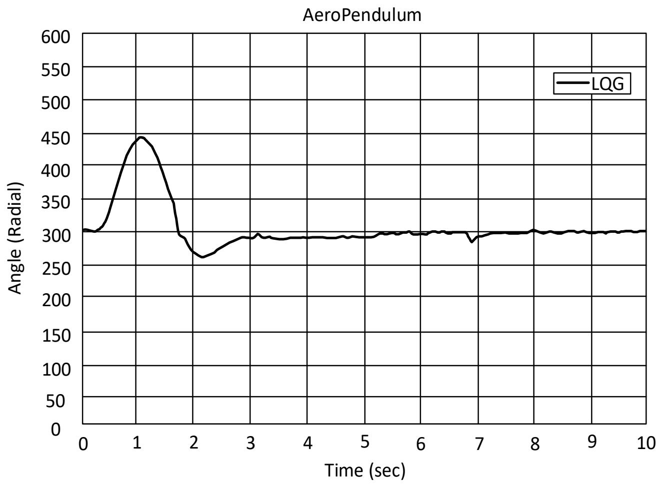

Fig. 13 Angle for Aero Pendulum with LQG controller.

In evaluating the Aero pendulum's simulation and control system, the experiment compared the LQR, LQI, and LQG controllers with the actual Aero pendulum structure. The angular values of the Aero pendulum were observed for stabilization, as indicated in Figure 14.

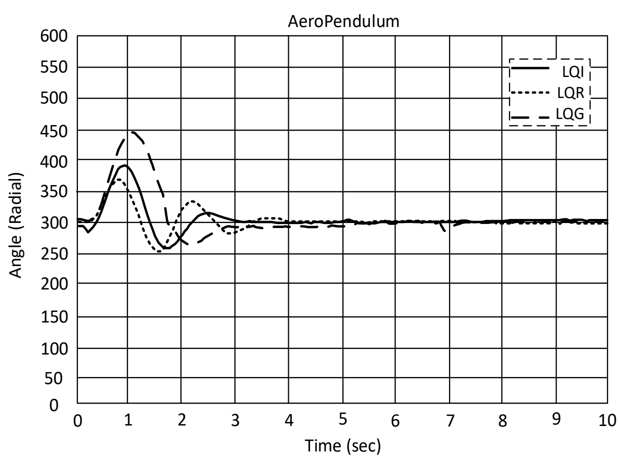

Fig. 14 Angle for Aero Pendulum with compare the 3 controller.

E. Comparison of performance index

According to the experimental results shown in Figure 14, the LQI controller successfully stabilizes the Aero pendulum, exhibiting higher effectiveness in attaining balance compared to the LQR and LQG controllers[14,15]. This thesis evaluates the balance robot control system's performance using the Integrated Error Square (ISE), which involves integrating the squared error over time.

$$ISE = \int_{0}^{\infty} e^2(t)\,dt \qquad (11)$$

One hundred randomly generated resolution values have been used to evaluate the performance of the LQR, LQI, and LQG control systems.

TABLE II
PERFORMANCE INDEX VALUE

Controller	VALUES
LQR Controller	0.733
LQI Controller	0.693
LQG Controller	0.742

V. CONCLUSION

The article summarizes the design of the LQR controller and can serve as a foundation for developing the LQI and LQG controllers. The article aims to ensure that the aero pendulum can maintain its balance. Three prototypes are required for this purpose. Understanding the mathematical principles of the aero pendulum is crucial before proceeding. The article in the thesis discusses these fundamental principles. Newton's aero pendulum encompasses the mathematics of the aero pendulum and considers the principles and factors for the design and science of the aero pendulum based on centralized kinetic energy.

Furthermore, the mathematics of the aero pendulum is rooted in an ideal Newtonian axis that prevents calculations, causing Pose in the mathematics of the aero pendulum. The article also delves into using the principles of the aero pendulum as the basis for the subsequent drive system. Considering the perspective and feeling that typically govern the aero pendulum, the article thoroughly examines the aero pendulum system. It compares the structural design components, assesses the angular performance of the aero pendulum with the body, elucidates the fundamental principles of the aero pendulum, and calculates the index value. The control system of the aero pendulum is scrutinized using LQR, LQI, and LQG control systems. Based on the aero pendulum's mathematics, this evaluation ensures that it maintains properties, allowing it to retain its balance.

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